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Factorial correlators: angular scaling within QCD jets

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Abstract. Factorial correlators measure the amount of dynamical correlation in the multiplicity between two separated phase-space windows. We present the analytical derivation of factorial correlators for a QCD jet described at the double logarithmic (DL) accuracy. We obtain a new angular scaling property for properly normalized correlators between two solid-angle cells or two rings around the jet axis. Normalized QCD factorial correlators scale with the angular distance and are independent of the window size. Scaling violations are expected beyond the DL approximation, in particular from the subjet structure. Experimental tests are feasible, and thus would be welcome.

1 Introduction

Large multiplicity fluctuations observed in high energy collisions have already been studied for many years [1]. Advanced methods of data analysis like, e.g., the factorial moment approach [2–4] have been introduced and implemented for the analysis of multiplicity patterns. Finally, these led to the discovery of *intermittency* in multiparticle production which refers to the scaling of factorial moments with the size of a single bin within the analyzed pattern [3,4].

Many different models have been proposed for the explanation of the effect [5]. Some suggested that an underlying final state multiparticle cascade may be responsible for the scaling of the particle moments [3,6]. Straightforward calculations performed for multiplicative random cascading models [2,3,7] led to qualitative predictions for the scaling behavior of factorial moments which were backed afterwards by analyses proposed in the framework of the standard theory of strong interactions (QCD) [8]. Monte Carlo simulations based on conventional QCD parton cascading tend to describe quite well this effect [9], confirming the relevance of scaling properties in QCD parton cascading. However, discrepancies in the precise comparison with *analytical* predictions remain [9].

So far, both the phenomenological and theoretical investigations of multiplicity patterns have concentrated mostly on different kinds of particle moments estimated for a single bin [5]. There are, however, still intriguing questions remaining about the properties of correlations between different bins. The observables related to these correlations are expected to reflect the presence of large dynamical fluctuations underlying the pattern, stronger than the averaged observables estimated for a single bin. To investigate these bin–bin correlations the factorial correlators [3,4] have been introduced.

Factorial correlators seem to contain some extra information on multiplicity fluctuations which may be used to complete and adjust the information obtained from the standard factorial moment analysis [10]. Moreover, the present status of experimental investigations [9,11] allows one to expect that the comparison of model predictions with real data will be possible soon. It could help to investigate more systematically the validity of QCD Monte Carlo approaches for the description of fluctuations in jets and to discuss, using a wider set of data, the relevance and problems of analytical QCD calculations.

However, factorial correlators have been studied only in the framework of phenomenological models [5]. The rigorous analysis in the framework of QCD (even at leading log orders) has not been performed so far.

This paper aims to fill the gap by presenting the analytical derivation of factorial correlators performed for the QCD parton cascade [8] at the double logarithmic (DL) accuracy [12]. For simplicity we consider only the fixed α_S case, expecting that it gives a good qualitative estimation of the scaling exponents as was realized previously for the case of the factorial moments [8]. The obtained scaling dependence of the correlators on the relative distance between the two solid-angle cells recovers a similar result obtained in the framework of the random cascading α model [2]. This seems to be a kind of universal relation.

This paper is organized as follows. In the next section we introduce factorial correlators defined for small solid-angle cells around the subjet direction. In Sect. 3, using

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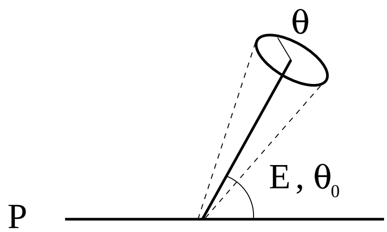

Fig. 1. Example of phase-space cell for the QCD parton cascade. The two-dimensional cone of half-opening angle θ is placed at both solid angle θ_0 and azimuthal angle ϕ taken with respect to the main jet axis

the DL generating functional [12], we derive the inclusive two gluon distribution which is necessary to evaluate the correlators. In Sect. 4 the leading contribution to the factorial correlator is estimated, and is found to obey a scaling law similar to the one in [2] for random multiplicative cascading models. In Sect. 5 we discuss briefly the modifications which could come from relaxing some of our approximations: including running α_S or energy momentum conservation. Finally, in Sect. 6 we sum up our results and present our conclusions, including suggestions for the experimental evaluation of the *normalized factorial correlators* and a discussion of the new QCD scaling law found in the DL approximation of QCD.

2 Factorial correlators in QCD jets

Normalized factorial moments F_q [2–5] designed to study multiplicity fluctuations in a given phase-space cell of size δ are defined by

$$F_q(\delta) = \frac{\langle n(n-1)\dots(n-q+1)\rangle_\delta}{\langle n\rangle_\delta^q}, \qquad (1)$$

where n is the particle multiplicity in the phase-space cell, and the average $\langle\rangle$ is made over events. Among other types of fluctuations studied using factorial moments, the intermittency regime corresponds to moments which scale with the size of the phase-space cell as

$$F_q(\delta) \sim \left(\frac{\Delta}{\delta}\right)^{\phi_q}, \qquad (2)$$

where Δ denotes the size of the whole available phase space, and ϕ_q is a positive scaling exponent known also as the intermittency exponent.

In order to study correlations between different phase-space cells one introduces factorial correlators F_{q_1,q_2} (known also as multivariate factorial moments) [2,5] which may be regarded as the multidimensional extension of the moments (1). They take the form

$$\begin{aligned} &F_{q_1,q_2}(\delta_1,\delta_2) \\ &= \Big\{ \Big(\langle n(n-1)\dots(n-q_1+1) \mid_{\delta_1} n(n-1) \\ &\quad \dots(n-q_2+1) \mid_{\delta_2} \rangle \Big) \Big/ \Big(\langle n\rangle_{\delta_1}^{q_1} \langle n\rangle_{\delta_2}^{q_2} \Big) \Big\}, \end{aligned} \qquad (3)$$

where δ_1 and δ_2 denote the sizes of two separate phase-space cells. Assuming a multiplicative random cascade underlying the particle production, one predicts a recursive relation between the scaling exponents for factorial moments and factorial correlators:

$$\phi_{q_1,q_2} = \phi_{q_1+q_2} - \phi_{q_1} - \phi_{q_2}, \qquad (4)$$

where ϕ_{q_1,q_2} is the intermittency exponent defined for doubly normalized factorial correlators which correspond to the factorial correlator (3) divided by factorial moments derived for δ_1 and δ_2 cells respectively:

$$\frac{F_{q_1,q_2}(\delta_1,\delta_2)}{F_{q_1}(\delta_1)F_{q_2}(\delta_2)} \sim \left(\frac{\Delta}{\delta_{12}}\right)^{\phi_{q_1,q_2}}, \qquad (5)$$

where δ_{12} is the relative distance between the two phase-space cells δ_1 and δ_2. Note the interesting feature that the dependence on the individual phase-space cells δ_1 and δ_2 disappears from the factorial correlators when they are normalized as in (5).

For the QCD parton cascade [8] the phase-space cell is more conveniently chosen to correspond to a window in the emission solid angle θ (see e.g. Fig. 1). The window size θ is then compared to a large scale θ_0 which denotes the jet emission angle. The window may be either a one-dimensional ring of aperture θ placed at the angle θ_0 with respect to the sphericity axis, or it may be a two-dimensional cone of half-opening angle θ placed at both the solid angle θ_0 and azimuthal angle ϕ taken with respect to the main jet axis.

It was found that for QCD angular factorial moments [8] there is a scaling relation:

$$F_q(\theta) \sim \left(\frac{\theta_0}{\theta}\right)^{\phi_q^{\mathrm{QCD}}}, \qquad (6)$$

where the intermittency exponent ϕ_q^{QCD} calculated for simplicity in double logarithmic approximation with fixed coupling constant α_S reads

$$\phi_q^{\mathrm{QCD}} = \frac{\gamma_0}{q} - \gamma_0 q + D(q-1). \qquad (7)$$

The number D denotes the window dimension, and it equals $D=1$ for the ring, and $D=2$ for the cone. The coefficient γ_0 is the QCD anomalous dimension for the gluon cascade which for fixed α_S equals $\gamma_0^2 = 4N_C(\alpha_S)/(2\pi)$ [12], where N_C denotes the number of colors.

In order to extend the intermittent analysis by investigating also the possible correlations between particle flows measured at two different rings around the subjet axis (E,θ_0) let us introduce the *angular factorial correlators* for the QCD parton cascade, defined as follows. Similarly as for the angular factorial moments one would identify the large scale (size of the whole available phase space Δ) with the respective subjet emission angle θ_0, and the small scales δ_1, δ_2 with the window apertures θ_1, θ_2 (cf. Figure 2). We will consider parton flows emitted into two rings placed at the separation angle θ_{12} with respect to the

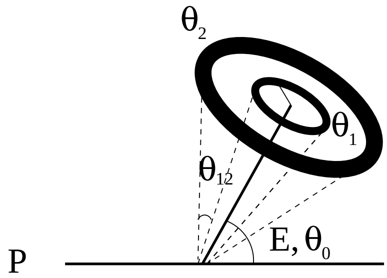

Fig. 2. Phase-space cells for angular factorial correlators. Parton flows are emitted into two rings placed at the separation angle θ_{12} with respect to the subjet axis. The ring openings are θ_1 and θ_2 respectively

subjet axis. The ring openings are θ_1 and θ_2 respectively (see Fig. 2). In using DL approximation framework, we have to assume that the angles are small with respect to the subjet direction. More precisely, we will assume that they obey the inequalities

$$\theta_1, \theta_2 \ll \theta_{12} \ll \theta_0. \tag{8}$$

Henceforth $\theta_{01} \sim \theta_{02}$, and the relative bin distance δ_{12} in the one-dimensional approximation then corresponds to the angular distance $\theta_{12} = \theta_{02} - \theta_{01}$ between the two rings. We will discuss the relevance of this DL approximation in our discussion in Sect. 5.

Having defined angular factorial correlators, we may now estimate them with a good accuracy from the convolution of the inclusive two-particle density $D^{(2)}(P; E, \theta_0; k_1, k_2, \theta_{12}, \theta_1, \theta_2)$ with the respective multiplicity moments in the phase-space cells θ_1 and θ_2. Using (as for the case of factorial moments [8]) their expression in the so-called KNO limit proportional up to constants to the q_1th and q_2th power of the mean multiplicities $N(k_1\theta_1)$, $N(k_2\theta_2)$, we obtain

$$\begin{aligned} F_{q_1 q_2}(\theta_0; \theta_{12}, \theta_1, \theta_2) &\propto \int^E \frac{\mathrm{d}k_1}{k_1} \int^E \frac{\mathrm{d}k_2}{k_2} \\ &\times D^{(2)}(P; E, \theta_0; k_1, k_2, \theta_{12}, \theta_1, \theta_2) N^{q_1}(k_1\theta_1) N^{q_2}(k_2\theta_2), \end{aligned} \tag{9}$$

where E denotes the energy of the subjet.

The mean multiplicity for the QCD parton cascade is dependent on an infrared cut-off μ, and it reads

$$N(k\theta) \sim \mathrm{e}^{\gamma_0 \ln(k\theta/\mu)}. \tag{10}$$

However, similarly as for the factorial moment case we expect that the cut-off dependence will disappear after normalization, i.e. when coming to normalized factorial correlators (5). The inclusive two-particle density $D^{(2)}(P; E, \theta_0; k_1, k_2, \theta_{12}, \theta_1, \theta_2)$ remains thus the only unknown quantity necessary to evaluate (9). Its explicit form will be derived in the next section.

3 Inclusive two-particle density

In order to obtain the explicit form of the two-particle density $D^{(2)}(P; E, \theta_0; k_1, k_2, \theta_{12}, \theta_1, \theta_2)$ to insert it into (9), we start with a derivation of a related quantity, $D_p^{(2)}(k_1, k_2)$, from the QCD parton cascade formalism.

The inclusive two-particle density $D_p^{(2)}(k_1, k_2)$ i.e. the inclusive density to produce two particles of momenta k_1, k_2 from a parent particle of momentum p may be derived in a convenient way from the generating functional for the QCD parton cascade [12]. This functional in DL approximation with fixed α_S takes the form

$$Z_p[u] = u(p) \mathrm{e}^{\int \mathrm{d}^3 k M_p(k)(Z_k[u]-1)}, \tag{11}$$

with the initial condition $Z_p[u] \mid_{\{u=1\}} = 1$. The function $u(p)$ is a probing function while the factor $M_p(k)$ describes the DL probability of emitting a particle of momentum k from a primary particle of momentum p. It reads

$$\mathrm{d}^3 k M_p(k) = \gamma_0^2 \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}k}{k} \frac{\phi}{2\pi} \bar{\theta}_p(k), \tag{12}$$

where the cut-off θ-function $\bar{\theta}_p(k)$ contains phase-space limitations resulting from a possible angular and energy ordering between parent particle p and child particle k:

$$\bar{\theta}_p(k) = \{p > k, \theta_{pk} < \theta_p, k\theta_{pk} > \mu\}, \tag{13}$$

where μ is the infrared cut-off.

The inclusive two-particle density is then defined as a functional derivative of $Z_p[u]$:

$$D_p^{(2)}(k_1, k_2) = k_1 k_2 \left. \frac{\delta^2 Z_p}{\delta u(k_1) \delta u(k_2)} \right|_{\{u=1\}}, \tag{14}$$

which results in a recursive equation for $D_p^{(2)}(k_1, k_2)$:

$$\begin{aligned} D_p^{(2)}(k_1, k_2) &= D_p^{(1)}(k_1) D_p^{(2)}(k_2) - \delta(\ln(p/k_1))\delta(\ln(p/k_2)) \\ &+ \int \mathrm{d}^3 k M_p(k) D_k^{(2)}(k_1, k_2), \end{aligned} \tag{15}$$

where $D_p^{(1)}(k_1) D_p^{(2)}(k_2)$ denotes the single particle inclusive densities for particles k_1 and k_2 respectively, and the integration limits are defined by (13) ($\max(\theta_1, \theta_2) < \theta < \theta_p$, $\max(k_1, k_2) < k < p$).

Taking into account the scaling properties of the DL phase-space measure (12), one obtains an equation for the modified inclusive two-particle density $D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12})$ with a scaling dependence on the relevant variables (chosen here to be $p \equiv E$, the energy of the subjet, and $k_1 > k_2$, $\theta_1 < \theta_2$):

$$\begin{aligned} D_p^{(2)}(k_1, k_2) &\Rightarrow D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12}) \\ &= D^{(1)}(E/k_1, \theta_0/\theta_1) D^{(1)}(E/k_2, \theta_0/\theta_2) \\ &- \delta(\ln(E/k_1))\delta(\ln(E/k_2)) \\ &+ \gamma_0^2 \int_{k_1}^{E} \frac{\mathrm{d}k}{k} \int_{\theta_{12}}^{\theta_0} \frac{\mathrm{d}\theta}{\theta} D^{(2)}(k/k_1, k/k_2, \theta/\theta_1, \theta/\theta_2; \theta_{12}), \end{aligned} \tag{16}$$

where we have assumed (cf. (15)) that the intermediate emissions represented by the homogeneous term of (15) *do not generate an additional particle flow into either the* θ_1

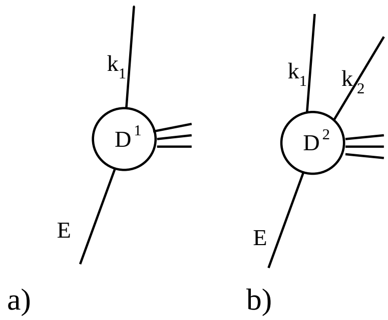

Fig. 3. Diagrammatic representation of the single particle inclusive density $D^{(1)}(E/k_1, \theta_0/\theta_1)$ **a** and the modified inclusive two-particle density $D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12})$ **b**

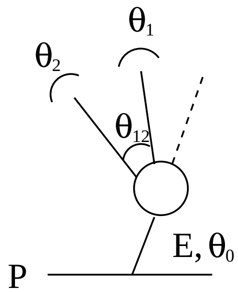

Fig. 4. Diagrammatic representation of the convolution $D_P^{(1,\mathrm{ex})}(E, \theta_0) \times D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12})$. It corresponds to the emission of particles from one subjet (E, θ_0) originating from the main jet (P, θ_P) into two rings of apertures θ_1, θ_2 with separation angle θ_{12}, placed around the subjet axis

or θ_2 *window* $(\theta_{12} < \theta < \theta_0)$. The densities $D^{(1)}(E/k_1, \theta_0/\theta_1)$ and $D^{(1)}(E/k_2, \theta_0/\theta_2)$ (cf. Fig. 3a) here denote single particle inclusive densities for particles k_1 and k_2 respectively. All notations are as in Fig. 2. The density $D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12})$ (cf. Fig. 3b) precisely represents (still in the one-dimensional approximation) the inclusive two-particle density to obtain two particles of energies k_1, k_2 from the subjet (E, θ_0) separated by the relative angle θ_{12}. The ring apertures are θ_1, θ_2 respectively.

The relation between $D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12})$ and the two-particle density (9) is the following:

$$
\begin{aligned}
&D^{(2)}(P; E, \theta_0; k_1, k_2, \theta_{12}, \theta_1, \theta_2) \\
&\quad = D_P^{(1,\mathrm{ex})}(E, \theta_0) \cdot D^{(2)}(E/k_1, E/k_2, \theta_0/\theta_1, \theta_0/\theta_2; \theta_{12}),
\end{aligned} \tag{17}
$$

where $D_P^{(1,\mathrm{ex})}(E, \theta_0)$ is the exclusive single particle density to produce a subjet of energy E placed at the opening angle θ_0 with respect to the main jet P.

To sum up, the convolution (17) represents the emission of particles from one subjet (E, θ_0) originating from the main jet (P, θ_P) into two rings of apertures θ_1, θ_2 with separation angle θ_{12}, placed around the subjet axis (see Fig. 4).

Introducing new variables:

$$
x_1 = \frac{k_1}{E}, \quad w_{12} = \frac{k_1}{k_2},
$$

$$
\begin{aligned}
&y_2 = \ln\frac{\theta_0}{\theta_{12}}, \quad t_1 = \ln\frac{\theta_{12}}{\theta_1}, \\
&t_2 = \ln\frac{\theta_{12}}{\theta_2},
\end{aligned} \tag{18}
$$

we rewrite (15) as

$$
\begin{aligned}
&D^{(2)}(1/x_1, w_{12}, t_1, t_2, y_2) \\
&\quad = D^{(1)}(1/x_1, y_2 + t_1) D^{(1)}(w_{12}/x_1, y_2 + t_2) \\
&\quad -\delta(\ln 1/x_1)\delta(\ln w_{12}/x_1) \\
&\quad +\gamma_0^2 \int_{x_1}^{1} \frac{\mathrm{d}x_1'}{x_1'} \int_0^{y_2} \mathrm{d}y_2' D^{(2)}(1/x_1', w_{12}, t_1, t_2, y_2').
\end{aligned} \tag{19}
$$

In order to solve (19), we transform it into moment space by means of the Mellin transform:

$$
\begin{aligned}
&D^{(2)}(\omega, w_{12}, t_1, t_2, y_2) \\
&\quad = \int_0^1 \mathrm{d}x_1 x_1^{\omega-1} D^{(2)}(1/x_1, w_{12}, t_1, t_2, y_2).
\end{aligned} \tag{20}
$$

Then we differentiate both sides of (19) with respect to y_2. Finally, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y_2} D^{(2)}(\omega, w_{12}, t_1, t_2, y_2) \\
&\quad = r(\omega, w_{12}, t_1, t_2, y_2) + \frac{\gamma_0^2}{\omega} D^{(2)}(\omega, w_{12}, t_1, t_2, y_2),
\end{aligned} \tag{21}
$$

where the function $r(\omega, w_{12}, t_{12}, y_2)$ reads

$$
\begin{aligned}
r(\omega, w_{12}, t_1, t_2, y_2) = &\int_0^1 \mathrm{d}x_1 x_1^{\omega-1} D^{(1)}(1/x_1, y_2 + t_1) \\
&\times D^{(1)}(w_{12}/x_1, y_2 + t_2).
\end{aligned} \tag{22}
$$

Equation (21) is an ordinary inhomogeneous linear differential equation. Taking into account the initial conditions defined by (19), its explicit solution takes the form

$$
\begin{aligned}
&D^{(2)}(\omega, w_{12}, t_1, t_2, y_2) \\
&\quad = r(\omega, w_{12}, t_1, t_2, y_2) + \frac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega) y_2} R(\omega, w_{12}, t_1, t_2, y_2) \\
&\quad -\delta(w_{12} - 1)\mathrm{e}^{(\gamma_0^2/\omega) y_2} - \frac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega) y_2} R(\omega, w_{12}, t_1, t_2, 0),
\end{aligned} \tag{23}
$$

where the function $r(\omega, w_{12}, t_1, t_2, y_2)$ was defined in (22), and the function $R(\omega, w_{12}, t_1, t_2, y_2)$ denotes the following indefinite integral of $r(\omega, w_{12}, t_{12}, y_2)$:

$$
R(\omega, w_{12}, t_1, t_2, y_2) = \int^{y_2} \mathrm{d}y \, r(\omega, w_{12}, t_1, t_2, y) \mathrm{e}^{-(\gamma_0^2/\omega) y_2}. \tag{24}
$$

4 The QCD factorial correlators: Derivation

As a function of the two-particle inclusive density (23), the convolution of $D^{(2)}$ and multiplicity correlators in (9) can now be expressed in explicit form. After a new change of variables:

$$l_1 = \log \frac{E}{k_1}, \quad s_{12} = \log \frac{k_1}{k_2}, \tag{25}$$

(9) rewritten for the *normalized angular correlators*,

$$\bar{F}_{q_1 q_2}(\theta_0/\theta_1, \theta_0/\theta_2) = \frac{F_{q_1 q_2}(\theta_0; \theta_{12}, \theta_1, \theta_2)}{D_P^{(1,\mathrm{ex})}(E, \theta_0) N^{q_1}(E\theta_0) N^{q_2}(E\theta_0)}, \tag{26}$$

takes the form

$$\begin{aligned}&\bar{F}_{q_1 q_2}(\theta_0/\theta_1, \theta_0/\theta_2)\\ &\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi i} \mathrm{e}^{\omega l_1}\\ &\times D^{(2)}(\omega, w_{12}, t_1, t_2, y_2)\\ &\times \mathrm{e}^{-q_1\gamma_0(y_2+t_1+l_1)} \mathrm{e}^{-q_2\gamma_0(y_2+t_2+l_1+s_{12})},\end{aligned} \tag{27}$$

where we substituted $D^{(2)}(1/x_1, w_{12}, t_1, t_2, y_2)$ by its inverse Mellin representation:

$$\begin{aligned}&D^{(2)}(1/x_1, w_{12}, t_1, t_2, y_2)\\ &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} (1/x_1)^\omega D^{(2)}(\omega, w_{12}, t_1, t_2, y_2).\end{aligned} \tag{28}$$

Now let us calculate term by term the contributions to the convolution integral (27) coming from the various components of the modified inclusive two-particle distribution (23) denoted (I), (II), (III) and (IV) as follows:

$$\begin{aligned}&D^{(2)}(\omega, w_{12}, t_1, t_2, y_2)\\ &= r(\omega, w_{12}, t_1, t_2, y_2) && \text{(I)}\\ &\quad + \tfrac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega) y_2} R(\omega, w_{12}, t_1, t_2, y_2) && \text{(II)}\\ &\quad - \delta(w_{12} - 1) \mathrm{e}^{(\gamma_0^2/\omega) y_2} && \text{(III)}\\ &\quad - \tfrac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega) y_2} R(\omega, w_{12}, t_1, t_2, 0) && \text{(IV)},\end{aligned} \tag{29}$$

These terms have the following Mellin representation:

$$\begin{aligned}\text{(I)}\quad & r(\omega, w_{12}, t_1, t_2, y_2)\\ &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi \mathrm{i}} \frac{1}{\omega - \omega_1 - \omega_2}\\ &\times \exp\left(\gamma_0^2 y_2 \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + \omega_2 s_{12}\right.\\ &\left. + \frac{\gamma_0^2}{\omega_1} t_1 + \frac{\gamma_0^2}{\omega_2} t_2\right),\end{aligned} \tag{30}$$

$$\begin{aligned}\text{(II)}\quad & \frac{\gamma_0^2}{\omega} \mathrm{e}^{\frac{\gamma_0^2}{\omega} y_2} R(\omega, w_{12}, t_1, t_2, y_2)\\ &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi \mathrm{i}} \frac{1}{\omega - \omega_1 - \omega_2}\\ &\times \frac{1}{\omega - \dfrac{\omega_1\omega_2}{\omega_1 + \omega_2}} \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \exp\left(\gamma_0^2 y_2 \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\right.\\ &\left. + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1} t_1 + \frac{\gamma_0^2}{\omega_2} t_2\right),\end{aligned} \tag{31}$$

$$\text{(III)}\quad \delta(w_{12} - 1) \mathrm{e}^{(\gamma_0^2/\omega) y_2} = \delta(s_{12}) \mathrm{e}^{(\gamma_0^2/\omega) y_2}, \tag{32}$$

$$\begin{aligned}\text{(IV)}\quad & \frac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega) y_2} R(\omega, w_{12}, t_1, t_2, 0)\\ &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi \mathrm{i}}\\ &\times \frac{1}{\omega - \omega_1 - \omega_2} \frac{1}{\omega - \dfrac{\omega_1 \omega_2}{\omega_1 + \omega_2}} \frac{\omega_1\omega_2}{\omega_1 + \omega_2}\\ &\times \exp\left(\frac{\gamma_0^2}{\omega} y_2 + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1} t_1 + \frac{\gamma_0^2}{\omega_2} t_2\right).\end{aligned} \tag{33}$$

The contributions of (30), (31), (32), (33) to (27) may be evaluated using the multidimensional saddle point approximation. For the first term of (30) the convolution (27) takes the form

$$\begin{aligned}&\bar{F}^I_{q_1 q_2}(\theta_0/\theta_1, \theta_0/\theta_2)\\ &\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi i} \mathrm{e}^{\omega l_1} r(\omega, w_{12}, t_1, t_2, y_2)\\ &\times \mathrm{e}^{-q_1\gamma_0(y_2+t_1+l_1)} \mathrm{e}^{-q_2\gamma_0(y_2+t_2+l_1+s_{12})}\\ &= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi \mathrm{i}}\\ &\times \frac{1}{\omega - \omega_1 - \omega_2} \exp\big(-q_1\gamma_0(y_2 + t_1 + l_1)\\ &-q_2\gamma_0(y_2 + t_2 + l_1 + s_{12}) + \omega l_1\big)\\ &\times \exp\left(\gamma_0^2 y_2 \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1} t_1 + \frac{\gamma_0^2}{\omega_2} t_2\right).\end{aligned} \tag{34}$$

After performing the integral over the ω-pole in (34) one obtains

$$\begin{aligned}&\bar{F}^I_{q_1 q_2}(\theta_0/\theta_1, \theta_0/\theta_2)\\ &\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi \mathrm{i}}\\ &\times \exp\left(S(\omega_1, \omega_2, l_1, s_{12}, t_1, t_2, y_2)\right),\end{aligned} \tag{35}$$

which in the saddle point approximation may be estimated to be (see (36) on top of the next page).

For (35), the function $S^I(\omega_1, \omega_2, l_1, s_{12}, t_1, t_2, y_2)$ reads

$$\begin{aligned}&S^I(\omega_1, \omega_2, l_1, s_{12}, t_1, t_2, y_2)\\ &= -q_1\gamma_0(y_2 + t_1 + l_1) - q_2\gamma_0(y_2 + t_2 + l_1 + s_{12})\\ &+(\omega_1 + \omega_2) l_1 + \gamma_0^2 y_2 \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\\ &+\omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1} t_1 + \frac{\gamma_0^2}{\omega_2} t_2.\end{aligned} \tag{37}$$

Hence after evaluating (35) one obtains

$$\begin{aligned}&\bar{F}^I_{q_1 q_2}(\theta_0/\theta_1, \theta_0/\theta_2)\\ &\sim -\frac{1}{4\pi^2} \exp\left\{y_2 \left(\frac{\gamma_0}{q_1} - \gamma_0 q_1 + \frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right.\\ &\left. + t_1 \left(\frac{\gamma_0}{q_1} - \gamma_0 q_1\right) + t_2 \left(\frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right\}.\end{aligned} \tag{38}$$

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi\mathrm{i}} \exp\left(S(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2)\right)$$
$$\sim \exp\left(S(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2)\,|_{\frac{\partial S}{\partial l_1}=0,\frac{\partial S}{\partial s_{12}}=0,\frac{\partial S}{\partial \omega_1}=0,\frac{\partial S}{\partial \omega_2}=0}\right)$$
$$\times \det\begin{pmatrix} \frac{\partial^2 S}{\partial l_1^2} & \frac{\partial^2 S}{\partial l_1 \partial s_{12}} & \frac{\partial^2 S}{\partial l_1 \partial \omega_1} & \frac{\partial^2 S}{\partial l_1 \partial \omega_2} \\ \frac{\partial^2 S}{\partial s_{12}\partial l_1} & \frac{\partial^2 S}{\partial s_{12}^2} & \frac{\partial^2 S}{\partial s_{12}\partial \omega_1} & \frac{\partial^2 S}{\partial s_{12}\partial \omega_2} \\ \frac{\partial^2 S}{\partial \omega_1 \partial l_1} & \frac{\partial^2 S}{\partial \omega_1 \partial s_{12}} & \frac{\partial^2 S}{\partial \omega_1^2} & \frac{\partial^2 S}{\partial \omega_1 \partial \omega_2} \\ \frac{\partial^2 S}{\partial \omega_2 \partial l_1} & \frac{\partial^2 S}{\partial \omega_2 \partial s_{12}} & \frac{\partial^2 S}{\partial \omega_2 \partial \omega_1} & \frac{\partial^2 S}{\partial \omega_2^2} \end{pmatrix}^{-1/2}_{\frac{\partial S}{\partial l_1}=0,\frac{\partial S}{\partial s_{12}}=0,\frac{\partial S}{\partial \omega_1}=0,\frac{\partial S}{\partial \omega_2}=0} . \quad (36)$$

For the second term (31), convolution (27) takes the form

$$\bar{F}^{II}_{q_1q_2}(\theta_0/\theta_1,\theta_0/\theta_2)$$
$$\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \times\frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \mathrm{e}^{\omega l_1} \frac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega)y_2} R(\omega,w_{12},t_1,t_2,y_2)$$
$$\times \mathrm{e}^{-q_1\gamma_0(y_2+t_1+l_1)}\mathrm{e}^{-q_2\gamma_0(y_2+t_2+l_1+s_{12})}$$
$$= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi\mathrm{i}}$$
$$\times \frac{1}{\omega-\omega_1-\omega_2} \frac{1}{\omega - \frac{\omega_1\omega_2}{\omega_1+\omega_2}} \frac{\omega_1\omega_2}{\omega_1+\omega_2}$$
$$\times \exp\left(-q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + \omega l_1\right) \quad (39)$$
$$\times \exp\left(\gamma_0^2 y_2 \left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2\right).$$

Since there are two ω-poles in (39) the integration over ω will give rise to two separate saddle point exponents:

$$\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \mathrm{e}^{\omega l_1} \frac{1}{\omega-\omega_1-\omega_2} \frac{1}{\omega-\frac{\omega_1\omega_2}{\omega_1+\omega_2}} \frac{\omega_1\omega_2}{\omega_1+\omega_2}$$
$$= \left(\mathrm{e}^{(\omega_1+\omega_2)l_1} - \mathrm{e}^{\omega_1\omega_2/(\omega_1+\omega_2)l_1}\right) \frac{\omega_1\omega_2}{\omega_1^2+\omega_1\omega_2+\omega_2^2}, \quad (40)$$

which have to be evaluated separately. One obtains

$$\bar{F}^{II}_{q_1q_2}(\theta_0/\theta_1,\theta_0/\theta_2)$$
$$\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi\mathrm{i}}$$
$$\times \left\{\exp(S^{\mathrm{IIa}}(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2)) - \exp(S^{\mathrm{IIb}}(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2))\right\}, \quad (41)$$

where S^{IIa}, S^{IIb} read

$$S^{\mathrm{IIa}}(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2)$$
$$= -q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12})$$
$$+(\omega_1+\omega_2)l_1 + \gamma_0^2 y_2\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) + \omega_2 s_{12}$$
$$+\frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2 + \ln\frac{\omega_1\omega_2}{\omega_1^2+\omega_1\omega_2+\omega_2^2}, \quad (42)$$

$$S^{\mathrm{IIb}}(\omega_1,\omega_2,l_1,s_{12},t_1,t_2,y_2)$$
$$= -q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12})$$
$$+\frac{\omega_1\omega_2}{\omega_1+\omega_2}l_1 + \gamma_0^2 y_2\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) + \omega_2 s_{12}$$
$$+\frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2 + \ln\frac{\omega_1\omega_2}{\omega_1^2+\omega_1\omega_2+\omega_2^2}. \quad (43)$$

Hence, after evaluating (39) one obtains

$$\bar{F}^{II}_{q_1q_2}(\theta_0/\theta_1,\theta_0/\theta_2) \quad (44)$$
$$\sim -\frac{1}{4\pi^2}\frac{q_1q_2}{q_1^2+q_1q_2+q_2^2}\left\{\exp\left[y_2\left(\frac{\gamma_0}{q_1}-\gamma_0 q_1 + \frac{\gamma_0}{q_2}-\gamma_0 q_2\right) + t_1\left(\frac{\gamma_0}{q_1}-\gamma_0 q_1\right) + t_2\left(\frac{\gamma_0}{q_2}-\gamma_0 q_2\right)\right]\right.$$
$$-\frac{(q_1+q_2)q_2}{q_1^2}\exp\left[y_2\left(\frac{\gamma_0}{q_1+q_2}-\gamma_0(q_1+q_2)\right)\right.$$
$$\left.\left.+\ t_1\left(-\gamma_0 q_1 - \frac{\gamma_0 q_1}{q_2(q_1+q_2)}\right) + t_2\left(\frac{\gamma_0}{q_2}-\gamma_0 q_2\right)\right]\right\}.$$

Similarly, for the third term (32) and fourth term (33), their convolutions (27) result in

$$\bar{F}^{\mathrm{III}}_{q_1q_2}(\theta_0/\theta_1,\theta_0/\theta_2)$$
$$\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \mathrm{e}^{\omega l_1} \delta(s_{12}) \mathrm{e}^{(\gamma_0^2/\omega)y_2}$$
$$\times \mathrm{e}^{-q_1\gamma_0(y_2+t_1+l_1)}\mathrm{e}^{-q_2\gamma_0(y_2+t_2+l_1+s_{12})}$$
$$= \int_0^\infty \mathrm{d}l_1 \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}p}{2\pi\mathrm{i}}$$
$$\times \exp\left(-q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + \omega l_1\right)$$
$$\times \exp\left(\frac{\gamma_0^2}{\omega}y_2 + p s_{12}\right), \quad (45)$$

$$\bar{F}^{\mathrm{IV}}_{q_1q_2}(\theta_0/\theta_1,\theta_0/\theta_2)$$
$$\sim \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \mathrm{e}^{\omega l_1} \frac{\gamma_0^2}{\omega} \mathrm{e}^{(\gamma_0^2/\omega)y_2}$$

$$\times R(\omega, w_{12}, t_1, t_2, 0)\mathrm{e}^{-q_1\gamma_0(y_2+t_1+l_1)}\mathrm{e}^{-q_2\gamma_0(y_2+t_2+l_1+s_{12})}$$
$$= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}s_{12} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\omega_2}{2\pi\mathrm{i}}$$
$$\times \frac{1}{\omega-\omega_1-\omega_2} \frac{1}{\omega - \frac{\omega_1\omega_2}{\omega_1+\omega_2}} \frac{\omega_1\omega_2}{\omega_1+\omega_2}$$
$$\times \exp\left(-q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + \omega l_1\right)$$
$$\times \exp\left(y_2\frac{\gamma_0^2}{\omega} + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2\right). \quad (46)$$

The respective saddle functions obtained similarly as in (40) and (41) are the following:

$$S^{\mathrm{III}}(\omega, p, l_1, s_{12}, t_1, t_2, y_2) = -q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + \omega l_1 + \frac{\gamma_0^2}{\omega}y_2 + p s_{12}, \quad (47)$$

$$S^{\mathrm{IVa}}(\omega_1, \omega_2, l_1, s_{12}, t_1, t_2, y_2) = -q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + (\omega_1+\omega_2)l_1 + \gamma_0^2 y_2 \frac{1}{\omega_1+\omega_2} + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2 + \ln\frac{\omega_1\omega_2}{\omega_1^2+\omega_1\omega_2+\omega_2^2}, \quad (48)$$

$$S^{\mathrm{IVb}}(\omega_1, \omega_2, l_1, s_{12}, t_1, t_2, y_2) = -q_1\gamma_0(y_2+t_1+l_1) - q_2\gamma_0(y_2+t_2+l_1+s_{12}) + \frac{\omega_1\omega_2}{\omega_1+\omega_2}l_1 + \gamma_0^2 y_2\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) + \omega_2 s_{12} + \frac{\gamma_0^2}{\omega_1}t_1 + \frac{\gamma_0^2}{\omega_2}t_2 + \ln\frac{\omega_1\omega_2}{\omega_1^2+\omega_1\omega_2+\omega_2^2}. \quad (49)$$

Hence after evaluating (45) and (46) one obtains

$$\bar{F}^{\mathrm{III}}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2) \sim -\frac{1}{4\pi^2}\exp\left\{y_2\left(\frac{\gamma_0}{q_1+q_2} - \gamma_0(q_1+q_2)\right) - t_1(\gamma_0 q_1) - t_2(\gamma_0 q_2)\right\}, \quad (50)$$

$$\bar{F}^{\mathrm{IV}}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2) \quad (51)$$
$$\sim -\frac{1}{4\pi^2}\frac{q_1q_2}{q_1^2+q_1q_2+q_2^2} \times \left\{\exp\left[y_2\left(\frac{\gamma_0}{q_1+q_2} - \gamma_0(q_1+q_2)\right) + t_1\left(\frac{\gamma_0}{q_1} - \gamma_0 q_1\right) + t_2\left(\frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right]\right.$$
$$\left. -\frac{(q_1+q_2)q_2}{q_1^2}\exp\left[y_2\left(\frac{\gamma_0}{q_1+q_2} - \gamma_0(q_1+q_2)\right) + t_1\left(-\gamma_0 q_1 - \frac{\gamma_0 q_1}{q_2(q_1+q_2)}\right) + t_2\left(\frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right]\right\}.$$

We now have to sum up the contributions (38), (44), (50) and (51) according to (27) and (29). Hence, finally, the *normalized angular correlators* (26) read

$$\bar{F}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2)$$
$$\sim \frac{1}{4\pi^2}A\exp\left\{y_2\left(\frac{\gamma_0}{q_1+q_2} - \gamma_0(q_1+q_2)\right) + t_1\left(\frac{\gamma_0}{q_1} - \gamma_0 q_1\right) + t_2\left(\frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right\}$$
$$+\frac{1}{4\pi^2}\exp\left\{y_2\left(\frac{\gamma_0}{q_1+q_2} - \gamma_0(q_1+q_2)\right) - t_1(\gamma_0 q_1) - t_2(\gamma_0 q_2)\right\}$$
$$-\frac{1}{4\pi^2}(A+1)\exp\left\{y_2\left(\frac{\gamma_0}{q_1} - \gamma_0 q_1 + \frac{\gamma_0}{q_2} - \gamma_0 q_2\right) + t_1\left(\frac{\gamma_0}{q_1} - \gamma_0 q_1\right) + t_2\left(\frac{\gamma_0}{q_2} - \gamma_0 q_2\right)\right\}, \quad (52)$$

where
$$A = \frac{q_1q_2}{q_1^2+q_1q_2+q_2^2}.$$

After dividing (52), in analogy to (5), by the product $F_{q_1}(\theta_0/\theta_1)F_{q_2}(\theta_0/\theta_2)$ one obtains

$$\frac{\bar{F}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2)}{F_{q_1}(\theta_0/\theta_1)F_{q_2}(\theta_0/\theta_2)} \quad (53)$$
$$\sim \frac{1}{4\pi^2}A\left(\frac{\theta_0}{\theta_{12}}\right)^{\phi_{q_1+q_2}-\phi_{q_1}-\phi_{q_2}} - \frac{1}{4\pi^2}(A+1) + \frac{1}{4\pi^2}\left(\frac{\theta_0}{\theta_{12}}\right)^{\phi_{q_1+q_2}-\phi_{q_1}-\phi_{q_2}}\left(\frac{\theta_1}{\theta_{12}}\right)^{\gamma_0/q_1}\left(\frac{\theta_2}{\theta_{12}}\right)^{\gamma_0/q_2},$$

where $\phi_{q_1+q_2} = \gamma_0/(q_1+q_2) - \gamma_0(q_1+q_2)$ and $\phi_{q_1} = \gamma_0/q_1 - \gamma_0 q_1$, $\phi_{q_2} = \gamma_0/q_2 - \gamma_0 q_2$. Note that the quantity $\phi_{q_1+q_2} - \phi_{q_1} - \phi_{q_2}$ has been denoted ϕ_{q_1,q_2} in formula (4).

5 Angular scaling of factorial correlators

In order to discuss the physical properties of the QCD factorial correlators, let us rewrite formula (52) in a different form:

$$\bar{F}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2)$$
$$\sim \frac{1}{4\pi^2}A\left\{\left(\frac{\theta_0}{\theta_{12}}\right)^{\phi_{q_1+q_2}}\left(\frac{\theta_{12}}{\theta_1}\right)^{\phi_{q_1}}\left(\frac{\theta_{12}}{\theta_2}\right)^{\phi_{q_2}} - \left(\frac{\theta_0}{\theta_1}\right)^{\phi_{q_1}}\left(\frac{\theta_0}{\theta_2}\right)^{\phi_{q_2}}\right\}$$
$$+\frac{1}{4\pi^2}\left\{\left(\frac{\theta_0}{\theta_{12}}\right)^{\phi_{q_1+q_2}}\left(\frac{\theta_{12}}{\theta_1}\right)^{\phi_{q_1}-\gamma_0/q_1}\left(\frac{\theta_{12}}{\theta_2}\right)^{\phi_{q_2}-\gamma_0/q_2} - \left(\frac{\theta_0}{\theta_1}\right)^{\phi_{q_1}}\left(\frac{\theta_0}{\theta_2}\right)^{\phi_{q_2}}\right\}. \quad (54)$$

The physical interpretation of the two brackets contributing to formula (54) is quite simple. Considering the first one, which is dominant at large values of $\theta_{12}/\theta_1, \theta_{12}/\theta_2$

(8): it corresponds to the contribution coming from the full development of the parton cascade (minus the value when $\theta_{12} \equiv \theta_0$, i.e. substracting the effect of cascading *before* θ_0). Indeed, due to the QCD constraints of angular ordering, the angular ordered path $\theta_0 \to \theta_{12}$ is populated by fluctuations with order $q_1 + q_2$, while the remaining separated paths from $\theta_{12} \to \theta_1$ and $\theta_{12} \to \theta_2$ correspond to the individual fluctuation patterns with order q_1 and q_2. In the QCD framework at DLA, this contribution is similar to the behavior of the random cascading models.

It is clear that this first term in (54) implies specific angular scaling properties of QCD jets (at DLA). Normalizing this term by the product $F_{q_1}(\theta_0/\theta_1)F_{q_2}(\theta_0/\theta_2)$ gives

$$\frac{\bar{F}_{q_1q_2}(\theta_0/\theta_1, \theta_0/\theta_2)}{F_{q_1}(\theta_0/\theta_1)F_{q_2}(\theta_0/\theta_2)} \sim \frac{1}{4\pi^2} A \left(\frac{\theta_0}{\theta_{12}}\right)^{\phi_{q_1+q_2}-\phi_{q_1}-\phi_{q_2}} . \tag{55}$$

The scaling properties of (55) can be expressed by the following three items:
(i) The normalized correlator (at DLA) depends only on the angular separation θ_{12}, and thus is independent of the window sizes θ_1, θ_2.
(ii) It obeys a scaling law as a function of the ratio θ_0/θ_{12}.
(iii) The scaling exponent is related to the ones of the factorial moments by $\phi_{q_1,q_2} = \phi_{q_1+q_2} - \phi_{q_1} - \phi_{q_2}$.

Such a prediction is similar to the one of random cascading models, which has previously [13] been discussed for soft hadronic multiproduction. In that case, the property (i) has been verified, while the dependence (ii) showed some bending and (iii) was largely violated since from the observation in some range of the resolution it appeared that $\phi_{q_1,q_2} \gg \phi_{q_1+q_2} - \phi_{q_1} - \phi_{q_2}$. We know now that the multiplicity fluctuations in soft hadronic multiproduction are influenced by Bose–Einstein enhancements. It would thus be interesting to measure by comparing the normalized correlators in jets, where the dynamics is more directly related to perturbative (and resummed) QCD properties. The experimental analysis can be done as an extension of what was done for angular factorial moments [9], where window rings around the jet axis have been considered as phase-space slices.

Interestingly enough, a second contribution appears in formula (54) which also has a simple physical interpretation. Contrary to the first term, the exponents $\phi_{q_1} - \gamma_0/q_1$ and $\phi_{q_1} - \gamma_0/q_1$ mean that the parton cascading structure during the second step of the process related to the separated paths from $\theta_{12} \to \theta_1$ and $\theta_{12} \to \theta_2$ is damped, since the corresponding fractal dimensions γ_0/q_1 and γ_0/q_1 are cancelled from the intermittency exponents. This corresponds to the probability of having a contribution of parton jets directly into the windows of observation. This contribution is obviously subdominant at DLA, since the exponents are smaller. It would lead to a violation of the scaling properties (i)–(iii).

However, it remains to be found whether, beyond the DLA approximation, such a contribution could be in practice larger than the first one. In particular, the lack of DLA exponentiation could be compensated by the strong contribution to multiplicity fluctuations of subjets directly hitting the observation windows. This study is beyond the scope of our paper devoted to the analysis of the DLA approximation but deserves our interest in the future.

6 Summary

We presented the analytical derivation of the factorial correlators performed for the QCD parton cascade at the double logarithmic (DL) accuracy. For simplicity we considered only the fixed α_S case, expecting that it gives a good qualitative estimation of the scaling exponents, as was realized previously for the factorial moments. The scaling dependence of the correlators on the relative distance between the two solid-angle cells recovered a similar result obtained in the framework of the random cascading α model [2,7], and seems to be a kind of universal relation.

However, it remains to be found whether the scaling holds also beyond the DLA approximation, where the contribution to multiplicity fluctuations coming from subjets directly hitting the observation windows may be dominant. This study is beyond the scope of our paper devoted to the analysis of the DLA approximation but remains for the future.

It would also be useful to compare these predictions with QCD Monte Carlo simulations (based on parton showers). It is already known that there is a noticeable difference between QCD predictions at DLA and QCD Monte Carlo simulations for factorial moments, these predictions being in better agreement (but not perfect) with the data. Since the origin of this discrepancy is not well understood at present, the study of factorial correlators could be useful for identifying the problem.

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